

Equal charge black holes and seven dimensional gauged supergravity

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Abstract

We present various supergravity black holes of different dimensions with some $U(1)$ charges set equal in a simple, common form. Black hole solutions of seven dimensional $U(1)^2$ gauged supergravity with three independent angular momenta and two equal $U(1)$ charges are obtained. We investigate the thermodynamics and the BPS limit of this solution, and find that there are rotating supersymmetric black holes without naked closed timelike curves. There are also supersymmetric topological soliton solutions without naked closed timelike curves that have a smooth geometry.

1 Introduction

There has recently been interest in solutions of gauged supergravity theories because of the AdS/CFT correspondence [1, 2, 3, 4]. Using exact solutions from the gravitational side of the correspondence, one hopes to learn about the dual gauge theory. Supersymmetric black hole solutions are useful for comparing the thermodynamics on both sides of the correspondence, since non-renormalization theorems mean that some results on the conformal field theory side may be extrapolated from weak to strong coupling. More general non-extremal black hole solutions have a non-zero Hawking temperature and are useful for studying the dual field theory at non-zero temperature.

It is of interest to consider not only four spacetime dimensions, but also higher spacetime dimensions $D > 4$, since the AdS/CFT correspondence may be studied in a variety of dimensions. Of particular interest are supergravity theories in spacetime dimensions $D = 4, 5, 7$, since the AdS/CFT correspondence relates these to superconformal field theories on a large number, N , of M2-branes, D3-branes, and M5-branes respectively; these branes preserve 32 supercharges and have maximal supersymmetry. More precisely, there is a duality between M-theory on $\text{AdS}_4 \times S^7$ and a non-abelian $D = 3$, $\mathcal{N} = 8$ superconformal theory, between type IIB string theory on $\text{AdS}_5 \times S^5$ and $D = 4$, $\mathcal{N} = 4$, $\text{SU}(N)$ super Yang–Mills theory, and between M-theory on $\text{AdS}_7 \times S^4$ and a non-abelian $D = 6$, $\mathcal{N} = (2, 0)$ superconformal theory. These respectively correspond to the maximal $D = 4$, $\mathcal{N} = 8$, $\text{SO}(8)$; $D = 5$, $\mathcal{N} = 8$, $\text{SO}(6)$; and $D = 7$, $\mathcal{N} = 4$, $\text{SO}(5)$ gauged supergravities, which have respective Cartan subgroups $\text{U}(1)^4$, $\text{U}(1)^3$ and $\text{U}(1)^2$.

There has been much progress over the last few years in obtaining new, non-extremal, asymptotically AdS black hole solutions of gauged supergravity theories in four [5], five [6, 7, 8, 9, 10] and seven [11] dimensions. However, in each case, we have not yet obtained a non-extremal solution with all rotation and charge parameters arbitrary. In addition to the non-extremal black hole families of solutions in the literature, there are also some known supersymmetric solutions, which should belong to a larger family of non-extremal solutions yet to be discovered. For the five dimensional case, building on previous work on supersymmetric AdS_5 black holes [12, 13], a supersymmetric solution with two rotation and three charge parameters arbitrary except for a single BPS constraint is known [14]. The basis for those studies is the classification of supersymmetric solutions using the G -structure formalism, introduced in the context of five dimensional minimal gauged supergravity by [15]. There has been some work which extends the supersymmetric classification to higher dimensions, for example in the dimension we focus on in this paper, $D = 7$ [16], however the classification becomes increasingly implicit. Nevertheless, supersymmetric solutions have been successfully found as a limit of non-extremal solutions without such a formalism, for example amongst the non-extremal solutions cited above.

In this paper, we consider certain supergravity black hole solutions that are both charged and rotating, but which have the simplification that we truncate to the Cartan subgroup, which is of the form $\text{U}(1)^k$, of the full gauge group, and some combinations of these $\text{U}(1)$ charges are set equal. The main result of this paper is a new black hole solution of seven dimensional gauged supergravity with three independent angular momenta and two equal $\text{U}(1)$ charges. However, for non-extremal solutions of gauged supergravity theories, there is no known solution generating technique, and instead one must rely on inspired guesswork. Therefore the “derivation” of the new solution will consist of a presentation of some previously known solutions in a unified manner, from which one might obtain a tight ansatz to find the result. In particular, in section 2, we review the relevant solutions of toroidally compactified heterotic supergravity in [17], of four dimensional gauged supergravity in [5], and

of five dimensional gauged supergravity in [7]. The case of toroidally compactified heterotic supergravity is useful as its Lagrangian gives rise to the ungauged limit of the supergravity Lagrangians we shall consider, and global symmetries of the theory give a mechanical solution generating technique that charges up a neutral solution, and so provide the ungauged limits of the solutions we seek. In section 3, we concentrate on the case of seven dimensional gauged supergravity, in particular reviewing the solution of [11], which has equal rotation parameters, specializing to the case of two equal $U(1)$ charges in the $U(1)^2$ maximal abelian subgroup of the full $SO(5)$ gauge group. Guided by these previous solutions with equal charges, we then write down a new solution of seven dimensional gauged supergravity with three independent angular momenta and both $U(1)$ charges equal. There is a brief discussion of the curvature singularities, which we find are similar to those of other higher dimensional black holes. We examine the thermodynamics of the solution and also the BPS limit, finding that the solution includes supersymmetric black holes without naked closed timelike curves. Like in four and five dimensions, these supersymmetric black holes must rotate. We investigate the fraction of supersymmetry preserved by the supersymmetric black holes and find that they are all $\frac{1}{8}$ supersymmetric. Of the more general supersymmetric solutions, naked closed timelike curves may also be avoided by a class of topological soliton solutions, which have a smooth geometry provided that the parameters obey a certain quantization condition. The topological soliton solutions are generally $\frac{1}{8}$ supersymmetric, but we investigate whether the supersymmetry can be enhanced, and find that it can in special cases.

2 Equal charge black holes

As a first step towards obtaining new seven dimensional gauged supergravity black holes with equal charges, we first reexamine some other supergravity black hole solutions with equal charges. We first consider solutions of ungauged supergravity, in particular those of toroidally compactified heterotic supergravity, before reviewing the most relevant solutions of four and five dimensional gauged supergravity.

2.1 Toroidally compactified heterotic supergravity

We first consider toroidally compactified heterotic supergravity, since its Lagrangian gives rise to the ungauged limit of the gauged supergravity theories we shall later consider. Solutions of this ungauged theory can be easier to obtain, and we shall later see examples of how certain solutions generalize to the gauged theories.

The bosonic fields of heterotic supergravity are the graviton g_{ab} , a dilaton ϕ , a Yang–Mills field A_a in the adjoint representation of $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$, and a Kalb–Ramond two-form potential B_{ab} . We truncate to the $U(1)^{16}$ Cartan subgroup of the Yang–Mills sector, which is a consistent bosonic truncation. In Einstein frame, the ten dimensional bosonic Lagrangian of heterotic supergravity is

$$\mathcal{L}_{10} = R \star 1 - \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} \sum_{I=1}^{16} e^{-\phi/2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} e^{-\phi} \star H_{(3)} \wedge H_{(3)}, \quad (2.1)$$

with $F_{(2)}^I = dA_{(1)}^I$ and $H_{(3)} = dB_{(2)} - \frac{1}{2} \sum_{I=1}^{16} F_{(2)}^I \wedge A_{(1)}^I$. Only terms with up to two derivatives appear in this Lagrangian.

If we perform a Kaluza–Klein reduction on a torus T^d to $D = 10 - d \geq 4$ dimensions, then we obtain the D -dimensional low-energy effective Lagrangian, which is of the form

$\mathcal{L}_D = \mathcal{L}_{D,1} + \mathcal{L}_{D,2} + \mathcal{L}_{D,3}$, where

$$\begin{aligned}
\mathcal{L}_{D,1} &= R \star 1 - \frac{1}{2} \left(\star d\phi \wedge d\phi + \sum_{i=1}^d \star d\phi_i \wedge d\phi_i + \sum_{i=1}^d e^{\mathbf{C}_i \cdot \phi} \star F_{i(2)} \wedge F_{i(2)} \right. \\
&\quad \left. + \sum_{1 \leq i < j \leq d} e^{\mathbf{C}_{ij} \cdot \phi} \star F_{ij(1)} \wedge F_{ij(1)} \right), \\
\mathcal{L}_{D,2} &= -\frac{1}{2} \sum_{I=1}^{16} \left(e^{\mathbf{A} \cdot \phi} \star F_{(2)}^I \wedge F_{(2)}^I + \sum_{i=1}^d e^{\mathbf{A}_i \cdot \phi} \star F_{i(1)}^I \wedge F_{i(1)}^I \right), \\
\mathcal{L}_{D,3} &= -\frac{1}{2} \left(e^{\mathbf{B} \cdot \phi} \star H_{(3)} \wedge H_{(3)} + \sum_{i=1}^d e^{\mathbf{B}_i \cdot \phi} \star H_{i(2)} \wedge H_{i(2)} \right. \\
&\quad \left. + \sum_{1 \leq i < j \leq d} e^{\mathbf{B}_{ij} \cdot \phi} \star H_{ij(1)} \wedge H_{ij(1)} \right). \tag{2.2}
\end{aligned}$$

$\mathcal{L}_{D,1}$ contains the ten dimensional scalar and terms that arise from reduction of the Einstein–Hilbert term, including d dilatons, d vectors and, from reduction of the vectors, $\frac{1}{2}d(d-1)$ so-called “axions”, which are scalars that arise from off-diagonal metric components. $\mathcal{L}_{D,2}$ comes from reduction of the two-form field strengths and $\mathcal{L}_{D,3}$ comes from reduction of the three-form field strength. $\phi = (\phi, \phi_1, \dots, \phi_d)$ is a vector field with its $d+1$ components being the ten dimensional scalar and the d dilatons. $\mathbf{A}, \mathbf{A}_i, \mathbf{B}, \mathbf{B}_i, \mathbf{B}_{ij}, \mathbf{C}_i$ and \mathbf{C}_{ij} are constant vectors with $d+1$ components, which are related to the root lattice of $O(10-D, 26-D)$. We shall not require the full expressions for the constant vectors or for the field strengths in terms of potentials, which may be found in [18] (see also [19] for the procedure applied to eleven dimensional supergravity), except for the relation $\mathbf{B} = 2\mathbf{A}$.

We shall consider the consistent bosonic truncation to the sector in which the only fields turned on are one linear combination of the dilatons, $\varphi = \mathbf{A} \cdot \phi \sqrt{(D-2)/2}$, two equal one-form potentials, $A_{(1)} = A_{(1)}^1 = A_{(1)}^2$, and the two-form potential $B_{(2)}$. The resulting field strengths are $F_{(2)} = dA_{(1)}$ and $H_{(3)} = dB_{(2)} - A_{(1)} \wedge dA_{(1)}$, in terms of which a Lagrangian for the field equations is

$$\mathcal{L}_D = R \star 1 - \frac{1}{2} \star d\varphi \wedge d\varphi - e^{2\varphi/\sqrt{2(D-2)}} \star F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{4\varphi/\sqrt{2(D-2)}} \star H_{(3)} \wedge H_{(3)}. \tag{2.3}$$

To make contact with other supergravity theories, we obtain an equivalent Lagrangian by dualizing the three-form field strength, $H_{(3)}$, in favour of a $(D-3)$ -form field strength, $F_{(D-3)}$. The Poincaré dualization procedure first involves noting the Bianchi identity for the three-form field strength, $dH_{(3)} + F_{(2)} \wedge F_{(2)} = 0$, which is imposed by adding to the Lagrangian the term $(-1)^{D-1} A_{(D-4)} \wedge (dH_{(3)} + F_{(2)} \wedge F_{(2)})$, treating $A_{(D-4)}$ as a Lagrange multiplier. Varying the modified Lagrangian with respect to $H_{(3)}$, the algebraic equation of motion for $H_{(3)}$ gives an expression for the dual field strength,

$$F_{(D-3)} = dA_{(D-4)} = e^{4\varphi/\sqrt{2(D-2)}} \star H_{(3)}. \tag{2.4}$$

Substituting back into the original Lagrangian and integrating by parts, we obtain the dual Lagrangian

$$\begin{aligned}
\mathcal{L}_D &= R \star 1 - \frac{1}{2} \star d\varphi \wedge d\varphi - e^{2\varphi/\sqrt{2(D-2)}} \star F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{-4\varphi/\sqrt{2(D-2)}} \star F_{(D-3)} \wedge F_{(D-3)} \\
&\quad + (-1)^{D-1} F_{(2)} \wedge F_{(2)} \wedge A_{(D-4)}, \tag{2.5}
\end{aligned}$$

where $F_{(2)} = dA_{(1)}$ and $F_{(D-3)} = dA_{(D-4)}$. The Chern–Simons terms appearing originate from the dependence of $H_{(3)}$ on $A_{(1)}$. We have chosen not to rescale $F_{(2)}$ to the canonical normalization for a single field, again for convenience when comparing with other supergravity theories.

2.2 Equal charge black holes of toroidally compactified heterotic supergravity

From uncharged black holes of Einstein gravity in the absence of a cosmological constant, one may generate charged generalizations in the context of toroidally compactified heterotic supergravity as a result of global symmetries of the theory. For rotating solutions in higher dimensions, the procedure was used in [17] to obtain charged and rotating black hole solutions from the neutral Myers–Perry solution [20]. We shall see that the charged solution simplifies substantially in the case that the charge parameters are equal, which will form the basis of generalizations to equal charge solutions of gauged supergravity theories.

The most illuminating way of writing the solution in the equal charge case is to use latitudinal and azimuthal coordinates that generalize the coordinates used for the Plebański solution [21] (see also [22] for the inclusion, in four dimensions, of an acceleration parameter, as in the C-metric), through which the solution takes a rather symmetrical form, and to write the metric using a set of simple vielbeins. In higher dimensions, such an approach was used in [23] to obtain NUT charge generalizations of the higher dimensional Kerr–AdS solution. There, the key observation was that for the round metric on a unit sphere S^{D-2} ,

$$d\Omega_{D-2}^2 = \sum_{i=1}^{\lfloor D/2 \rfloor} d\mu_i^2 + \sum_{i=1}^{\lfloor (D-1)/2 \rfloor} \mu_i^2 d\phi_i^2, \quad \sum_{i=1}^{\lfloor D/2 \rfloor} \mu_i^2 = 1, \quad (2.6)$$

the latitudinal coordinates μ_i may be parameterized as

$$\mu_i^2 = \frac{\prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2)}{\prod'_{k=1}^n (a_i^2 - a_k^2)}, \quad (2.7)$$

where $D = 2n$ for even dimensions and $D = 2n + 1$ for odd dimensions. We have used the notation \prod' to indicate that we exclude the factor that vanishes from a product. The round metric then takes a diagonal form with

$$\sum_{i=1}^n d\mu_i^2 = (-1)^{n+1} \sum_{\alpha=1}^{n-1} \frac{y_\alpha^2 \prod'_{\beta=1}^{n-1} (y_\alpha^2 - y_\beta^2)}{\prod_{k=1}^n (a_k^2 - y_\alpha^2)} dy_\alpha^2. \quad (2.8)$$

For the azimuthal coordinates, the higher dimensional generalization of Boyer–Lindquist coordinates ϕ_i retain the property that they are periodic with canonical normalization so that their period is 2π . For computational purposes and conciseness, it is more convenient to perform a linear coordinate transformation of the azimuthal coordinates ϕ_i and the Boyer–Lindquist time coordinate t to a higher dimensional generalization of those used by Plebański, as we shall later use and denote by ψ_i ; the coordinate change may be found in [23]. Readers unfamiliar with this coordinate system for higher dimensional black holes may find it helpful to look at more specific examples first, before considering arbitrary dimensions, for example in [23], where explicit expressions for the Kerr–NUT–AdS solution in six and seven dimensions may be found, and the solutions of four and five dimensional gauged supergravity that we review later.

The latitudinal coordinates y_α were first introduced by Jacobi [24], so I suggest calling them Jacobi coordinates (although the $n = 3$ case was previously considered by Neumann in analysing the three dimensional harmonic oscillator constrained on S^2 [25]). Because of the use of these azimuthal coordinates by Carter for expressing the Kerr–AdS solution [26], I suggest that the ψ_i coordinates be called Carter coordinates. Although Plebański [21] suggested the name of Boyer coordinates for the full set of all four coordinates, such terminology has not caught on, perhaps because of possible confusion with Boyer–Lindquist coordinates. I therefore instead suggest that the full set of coordinates (y_α, ψ_i) be called Jacobi–Carter coordinates.

We should note that there are some typographical errors in the general solution of [17], noted for example in [11]. Also, compared to [17], we have changed the sign of ϕ_i and set $l_i = a_i$.

2.2.1 Even dimensions $D = 2n$

In Boyer–Lindquist coordinates, the solution of [17] with both U(1) charges set equal in even dimensions $D = 2n$ may be written as

$$\begin{aligned} ds^2 = H^{2/(D-2)} & \left\{ -\frac{R}{H^2 U} \mathcal{A}^2 + \frac{U}{R} dr^2 \right. \\ & \left. + \sum_{\alpha=1}^{n-1} \left[\frac{X_\alpha}{U_\alpha} \left(dt - \sum_{i=1}^{n-1} \frac{(r^2 + a_i^2) \gamma_i}{a_i^2 - y_\alpha^2} d\tilde{\phi}_i - \frac{2ms^2 r}{HU} \mathcal{A} \right)^2 + \frac{U_\alpha}{X_\alpha} dy_\alpha^2 \right] \right\}, \\ e^{\varphi'} = \frac{1}{H}, \quad A_{(1)} = \frac{2mscr}{HU} \mathcal{A}, \quad B_{(2)} = \frac{2ms^2 r}{HU} dt \wedge \sum_{i=1}^{n-1} \gamma_i d\tilde{\phi}_i, \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} U &= \prod_{\alpha=1}^{n-1} (r^2 + y_\alpha^2), \quad U_\alpha = -(r^2 + y_\alpha^2) \prod'_{\beta=1}^{n-1} (y_\beta^2 - y_\alpha^2), \quad \gamma_i = \prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2), \\ R &= \prod_{k=1}^{n-1} (r^2 + a_k^2) - 2mr, \quad X_\alpha = -\prod_{k=1}^{n-1} (a_k^2 - y_\alpha^2), \quad \mathcal{A} = dt - \sum_{i=1}^{n-1} \gamma_i d\tilde{\phi}_i, \\ H &= 1 + \frac{2ms^2 r}{U}, \quad s = \sinh \delta, \quad c = \cosh \delta, \quad \tilde{\phi}_i = \frac{\phi_i}{a_i \prod'_{k=1}^{n-1} (a_i^2 - a_k^2)}, \end{aligned} \quad (2.10)$$

and the normalization of the scalar of [17], φ' , is related to the scalar that we have been using, φ , by $\varphi' = \sqrt{(D-2)/2} \varphi$.

The metric takes a slightly simpler form if we make a linear coordinate transformation of the azimuthal coordinates ϕ_i to Carter coordinates ψ_i . In these coordinates, there is a compact expression for the solution if we make analytic continuations to give a Riemannian metric. We analytically continue the radial coordinate r , so that it appears on an equal footing as the other coordinates y_α , and define n coordinates x_μ by

$$\begin{aligned} x_\alpha &= y_\alpha, \quad 1 \leq \alpha \leq n-1, \\ x_n &= ir. \end{aligned} \quad (2.11)$$

To keep the metric real, we also make the analytic continuation $m_n = -im$. The new coordinate $t' = \psi_0$, which would be a time coordinate in Lorentzian signature, may be placed

on a similar footing as the latitudinal coordinates ψ_i . It is also convenient to record the dual potential $A_{(D-4)}$ rather than $B_{(2)}$. The solution takes the form

$$\begin{aligned} ds^2 &= H^{2/(D-2)} \sum_{\mu=1}^n \left[\frac{X_\mu}{U_\mu} \left(\mathcal{A}_\mu - \frac{2m_n s^2 x_n}{H U_n} \mathcal{A}_n \right)^2 + \frac{U_\mu}{X_\mu} dx_\mu^2 \right], \\ e^{\varphi'} &= \frac{1}{H}, \quad A_{(1)} = \frac{2m_n s c x_n}{H U_n} \mathcal{A}_n, \\ A_{(D-4)} &= \frac{2im_n s^2 \prod_{\alpha=1}^{n-1} x_\alpha}{(n-2)! U_n} \left(\sum_{\alpha=1}^{n-1} \frac{x_\alpha^2 - x_n^2}{x_\alpha} dx_\alpha \wedge \mathcal{A}_{\alpha n} \right)^{n-2}, \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} U_\mu &= \prod_{\nu=1}^n (x_\nu^2 - x_\mu^2), \quad X_\mu = - \prod_{k=1}^{n-1} (a_k^2 - x_\mu^2) + 2m_\mu x_\mu, \quad m_\mu = m_n \delta_{\mu n}, \\ \mathcal{A}_\mu &= \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k, \quad A_\mu^{(k)} = \sum_{\substack{\nu_1 < \nu_2 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \\ \mathcal{A}_{\mu\nu} &= \sum_{k=1}^{n-1} A_{\mu\nu}^{(k-1)} d\psi_k, \quad A_{\mu\nu}^{(k)} = \sum_{\substack{\nu_1 < \nu_2 < \dots < \nu_k \\ \nu_i \neq \mu, \nu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \\ H &= 1 + \frac{2m_n s^2 x_n}{U_n}, \quad s = \sinh \delta, \quad c = \cosh \delta. \end{aligned} \quad (2.13)$$

2.2.2 Odd dimensions $D = 2n + 1$

In Boyer–Lindquist coordinates, the solution of [17] with both U(1) charges set equal in odd dimensions $D = 2n + 1$ may be written as

$$\begin{aligned} ds^2 &= H^{2/(D-2)} \left\{ - \frac{R}{H^2 U} \mathcal{A}^2 + \frac{U}{R} dr^2 \right. \\ &\quad + \sum_{\alpha=1}^{n-1} \left[\frac{X_\alpha}{U_\alpha} \left(dt - \sum_{i=1}^n \frac{(r^2 + a_i^2) \gamma_i}{a_i^2 - y_\alpha^2} d\tilde{\phi}_i - \frac{2ms^2}{H U} \mathcal{A} \right)^2 + \frac{U_\alpha}{X_\alpha} dy_\alpha^2 \right] \\ &\quad \left. + \frac{\prod_{k=1}^n a_k^2}{r^2 \prod_{\alpha=1}^{n-1} y_\alpha^2} \left(dt - \sum_{i=1}^n \frac{(r^2 + a_i^2) \gamma_i}{a_i^2} d\tilde{\phi}_i - \frac{2ms^2}{H U} \mathcal{A} \right)^2 \right\}, \\ e^{\varphi'} &= \frac{1}{H}, \quad A_{(1)} = \frac{2msc}{H U} \mathcal{A}, \quad B_{(2)} = \frac{2ms^2}{H U} dt \wedge \sum_{i=1}^n \gamma_i d\tilde{\phi}_i, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} U &= \prod_{\alpha=1}^{n-1} (r^2 + y_\alpha^2), \quad U_\alpha = -(r^2 + y_\alpha^2) \prod_{\beta=1}^{n-1} (y_\beta^2 - y_\alpha^2), \quad \gamma_i = a_i^2 \prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2), \\ R &= \frac{1}{r^2} \prod_{k=1}^n (r^2 + a_k^2) - 2m, \quad X_\alpha = \frac{1}{y_\alpha^2} \prod_{k=1}^n (a_k^2 - y_\alpha^2), \quad \mathcal{A} = dt - \sum_{i=1}^n \gamma_i d\tilde{\phi}_i, \\ H &= 1 + \frac{2ms^2}{U}, \quad s = \sinh \delta, \quad c = \cosh \delta, \quad \tilde{\phi}_i = \frac{\phi_i}{a_i \prod_{k=1}^n (a_i^2 - a_k^2)}, \end{aligned} \quad (2.15)$$

again with $\varphi' = \sqrt{(D-2)/2}\varphi$.

We again analytically continue the radial coordinate r for convenience when using Carter coordinates, using the same definition for the n coordinates x_μ as for the even dimensional case in (2.11). The solution after these analytic continuations is

$$\begin{aligned} ds^2 &= H^{2/(D-2)} \left\{ \sum_{\mu=1}^n \left[\frac{X_\mu}{U_\mu} \left(\mathcal{A}_\mu - \frac{2m_n s^2}{H U_n} \mathcal{A}_n \right)^2 + \frac{U_\mu}{X_\mu} dx_\mu^2 \right] \right. \\ &\quad \left. - \frac{\prod_{i=1}^n a_i^2}{\prod_{\mu=1}^n x_\mu^2} \left(\sum_{k=0}^n A^{(k)} d\psi_k - \frac{2m_n s^2}{H U_n} \mathcal{A}_n \right)^2 \right\}, \\ e^{\varphi'} &= \frac{1}{H}, \quad A_{(1)} = \frac{2m_n s c}{H U_n} \mathcal{A}_n, \\ A_{(D-4)} &= \frac{2m_n s^2 \prod_{i=1}^n a_i}{(n-2)! U_n} \sum_{k=1}^n A_n^{(k-1)} d\psi_k \wedge \left(\sum_{\alpha=1}^{n-1} \frac{x_\alpha^2 - x_n^2}{x_\alpha} dx_\alpha \wedge \mathcal{A}_{\alpha n} \right)^{n-2}, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} U_\mu &= \prod_{\nu=1}^n (x_\nu^2 - x_\mu^2), \quad X_\mu = \frac{1}{x_\mu^2} \prod_{k=1}^n (a_k^2 - x_\mu^2) + 2m_\mu, \quad m_\mu = m_n \delta_{\mu n}, \\ A^{(k)} &= \sum_{\nu_1 < \nu_2 < \dots < \nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \end{aligned} \quad (2.17)$$

$$\begin{aligned} \mathcal{A}_\mu &= \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k, \quad A_\mu^{(k)} = \sum_{\substack{\nu_1 < \nu_2 < \dots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \\ \mathcal{A}_{\mu\nu} &= \sum_{k=1}^{n-1} A_{\mu\nu}^{(k-1)} d\psi_k, \quad A_{\mu\nu}^{(k)} = \sum_{\substack{\nu_1 < \nu_2 < \dots < \nu_k \\ \nu_i \neq \mu, \nu}} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_k}^2, \end{aligned}$$

$$H = 1 + \frac{2m_n s^2}{U_n}, \quad s = \sinh \delta, \quad c = \cosh \delta. \quad (2.18)$$

2.3 Four dimensional gauged supergravity

The maximal $D = 4$, $\mathcal{N} = 8$, $\text{SO}(8)$ gauged supergravity may be obtained by dimensional reduction of eleven dimensional supergravity on S^7 . Truncating so that we only include gauge fields in the $\text{U}(1)^4$ Cartan subgroup of the full gauge group, we arrive at $\mathcal{N} = 2$ gauged supergravity coupled to three vector multiplets. The bosonic fields are the graviton, four $\text{U}(1)$ gauge fields, three dilatons and three axions.

Considering the truncation to pairwise equal charges, the bosonic Lagrangian is

$$\begin{aligned} \mathcal{L}_4 &= R \star 1 - \frac{1}{2} \star d\varphi \wedge d\varphi - \frac{1}{2} e^{-2\varphi} \star d\chi \wedge d\chi - \frac{1}{2} e^\varphi (\star F_{(2)}^1 \wedge F_{(2)}^1 + \star F_{(2)}^2 \wedge F_{(2)}^2) \\ &\quad - \frac{1}{2} \chi (F_{(2)}^1 \wedge F_{(2)}^1 + F_{(2)}^2 \wedge F_{(2)}^2) + g^2 (4 + 2 \cosh \varphi + e^{-\varphi} \chi^2) \star 1. \end{aligned} \quad (2.19)$$

Compared with [5], we have changed the sign of φ , and adjusted the sign of the potential so that setting both scalars to zero gives a negative cosmological constant.

Black hole solutions of this truncation of the gauged supergravity theory were obtained in [5]. They are parameterized by the angular momentum, two independent $\text{U}(1)$ charges,

mass and NUT charge, although we do not consider NUT charge here. In the ungauged limit, the solutions can be obtained from a more general known four charge solution by setting the charges to be pairwise equal.

A further consistent bosonic truncation is to take $A_{(1)}^1 = 0$. The field equations can be obtained from the Lagrangian

$$\begin{aligned} \mathcal{L}_4 = & R \star 1 - \frac{1}{2} \star d\varphi \wedge d\varphi - X^{-2} \star F_{(2)} \wedge F_{(2)} - \frac{1}{2} X^4 \star d\chi \wedge d\chi - \chi F_{(2)} \wedge F_{(2)} \\ & + g^2 (X^2 + 4 + X^{-2} + X^2 \chi^2) \star 1, \end{aligned} \quad (2.20)$$

where $X = e^{-\varphi/2}$ and $F_{(2)} = dA_{(1)} = (1/\sqrt{2})dA_{(1)}^2$. Setting $g = 0$, we recover the Lagrangian of (2.5).

The black hole solution without NUT charge of this further truncated sector that is relevant for us is, in Jacobi–Carter coordinates,

$$\begin{aligned} ds^2 = & H \left[\frac{r^2 + y^2}{R} dr^2 + \frac{r^2 + y^2}{Y} dy^2 - \frac{R}{H^2(r^2 + y^2)} \mathcal{A}^2 \right. \\ & \left. + \frac{Y}{r^2 + y^2} \left(dt' - r^2 d\psi_1 - \frac{qr}{H(r^2 + y^2)} \mathcal{A} \right)^2 \right], \\ X = & H^{-1/2}, \quad A_{(1)}^1 = 0, \quad A_{(1)}^2 = \frac{2mscr}{H(r^2 + y^2)} \mathcal{A}, \quad \chi = \frac{qy}{r^2 + y^2}, \end{aligned} \quad (2.21)$$

where

$$\begin{aligned} R = & (1 + g^2 r^2)(r^2 + a^2) + qg^2 r(2r^2 + a^2) + q^2 g^2 r^2 - 2mr, \\ Y = & (1 - g^2 y^2)(a^2 - y^2), \quad \mathcal{A} = dt + y^2 d\psi_1, \\ H = & 1 + \frac{qr}{r^2 + y^2}, \quad q = 2ms^2, \quad s = \sinh \delta, \quad c = \cosh \delta. \end{aligned} \quad (2.22)$$

Taking $g = 0$, we recover the solution of (2.12) for $D = 4$. Viewed as a solution of the $U(1)^4$ supergravity theory, in the notation of [5], we have taken the four charge parameters to be $\delta_1 = \delta_3 = 0$, $\delta_2 = \delta_4 = \delta$.

2.4 Five dimensional gauged supergravity

Performing a Kaluza–Klein reduction of type IIB supergravity on S^5 leads to $D = 5$, $\mathcal{N} = 8$, $SO(6) \cong SU(4)$ gauged supergravity. We truncate to include only gauge fields in the $U(1)^3$ Cartan subgroup of the full gauge group. There is a consistent truncation to minimal $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets. The bosonic fields are a graviton, three $U(1)$ gauge fields and two scalars.

The Lagrangian is

$$\mathcal{L}_5 = R \star 1 - \frac{1}{2} \sum_{i=1}^2 \star d\varphi_i \wedge d\varphi_i - \frac{1}{2} \sum_{I=1}^3 X_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I + 4g^2 \sum_{I=1}^3 X_I^{-1} \star 1 + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(1)}^3, \quad (2.23)$$

where

$$X_1 = e^{-\varphi_1/\sqrt{6}-\varphi_2/\sqrt{2}}, \quad X_2 = e^{-\varphi_1/\sqrt{6}+\varphi_2/\sqrt{2}}, \quad X_3 = e^{2\varphi_1/\sqrt{6}}, \quad F_{(2)}^I = dA_{(1)}^I. \quad (2.24)$$

We may perform a consistent bosonic truncation to the sector with $A_{(1)}^1 = A_{(1)}^2$ and $X = X_1 = X_2 = X_3^{-1/2} = e^{-\varphi_1/\sqrt{6}}$. Relabelling $\varphi_1 \rightarrow \varphi$, the bosonic field equations can be obtained from the Lagrangian

$$\begin{aligned} \mathcal{L}_5 = & R \star 1 - \frac{1}{2} \star d\varphi \wedge d\varphi - X^{-2} \star F_{(2)}^1 \wedge F_{(2)}^1 - \frac{1}{2} X^4 \star F_{(2)}^3 \wedge F_{(2)}^3 \\ & + 4g^2(2X^{-1} + X^2) \star 1 + F_{(2)}^1 \wedge F_{(2)}^1 \wedge A_{(1)}^3. \end{aligned} \quad (2.25)$$

Setting $g = 0$, we recover the Lagrangian of (2.5).

For our purposes, we consider the black hole solution of [7] that has two independent rotation parameters, two charges set equal, and the third charge set to a particular value once the other charges are fixed, so there is one independent charge parameter. The solution in Jacobi–Carter coordinates is

$$\begin{aligned} ds^2 = & H^{2/3} \left[\frac{r^2 + y^2}{R} dr^2 + \frac{r^2 + y^2}{Y} dy^2 - \frac{R}{H^2(r^2 + y^2)} \mathcal{A}^2 \right. \\ & + \frac{Y}{r^2 + y^2} \left(dt' - r^2 d\psi_1 - \frac{q}{H(r^2 + y^2)} \mathcal{A} \right)^2 \\ & \left. + \frac{a^2 b^2}{r^2 y^2} \left(dt' + (y^2 - r^2) d\psi_1 - r^2 y^2 d\psi_2 - \frac{q}{H(r^2 + y^2)} \mathcal{A} \right)^2 \right], \\ X_1 = X_2 = & H^{-1/3}, \quad X_3 = H^{2/3}, \\ A_{(1)}^1 = A_{(1)}^2 = & \frac{2msc}{H(r^2 + y^2)} \mathcal{A}, \quad A_{(1)}^3 = \frac{qab}{r^2 + y^2} (d\psi_1 + y^2 d\psi_2), \end{aligned} \quad (2.26)$$

where

$$\begin{aligned} R = & \frac{(1 + g^2 r^2)(r^2 + a^2)(r^2 + b^2)}{r^2} + qg^2(2r^2 + a^2 + b^2) + q^2 g^2 - 2m, \\ Y = & -\frac{(1 - g^2 y^2)(a^2 - y^2)(b^2 - y^2)}{y^2}, \quad \mathcal{A} = dt' + y^2 d\psi_1, \\ H = & 1 + \frac{q}{r^2 + y^2}, \quad q = 2ms^2, \quad s = \sinh \delta, \quad c = \cosh \delta. \end{aligned} \quad (2.27)$$

Taking $g = 0$, we recover the solution of (2.16) for $D = 5$.

A generalization of this solution has recently been discovered such that although there are two equal U(1) charges, the third may be independently specified [10]; this generalization also includes the five dimensional minimal gauged supergravity black hole solution of [8]. However, as this more general solution has two independent charge parameters rather than one, we do not require any of its additional features to guide us towards the new seven dimensional solution obtained in the next section.

3 Seven dimensional gauged supergravity black holes

Reducing eleven dimensional supergravity on S^4 leads to $D = 7$, $\mathcal{N} = 4$, SO(5) gauged supergravity. We truncate to include only gauge fields in the U(1)² Cartan subgroup of the full gauge group. The bosonic fields are a graviton, a three-form potential, two U(1) gauge fields and two scalars.

The bosonic Lagrangian is

$$\begin{aligned}\mathcal{L}_7 = & R \star 1 - \frac{1}{2} \sum_{i=1}^2 \star d\varphi_i \wedge d\varphi_i - \frac{1}{2} \sum_{I=1}^2 X_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} X_1^2 X_2^2 \star F_{(4)} \wedge F_{(4)} \\ & + 2g^2(8X_1 X_2 + 4X_1^{-1} X_2^{-2} + 4X_1^{-2} X_2^{-1} - X_1^{-4} X_2^{-4}) \star 1 \\ & + gF_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)},\end{aligned}\quad (3.1)$$

where

$$X_1 = e^{-\varphi_1/\sqrt{10}-\varphi_2/\sqrt{2}}, \quad X_2 = e^{-\varphi_1/\sqrt{10}+\varphi_2/\sqrt{2}}, \quad F_{(2)}^I = dA_{(1)}^I, \quad F_{(4)} = dA_{(3)}. \quad (3.2)$$

The resulting Einstein equation is

$$\begin{aligned}G_{ab} = & \sum_{i=1}^2 \left(\frac{1}{2} \nabla_a \varphi_i \nabla_b \varphi_i - \frac{1}{4} \nabla^c \varphi_i \nabla_c \varphi_i g_{ab} \right) + \sum_{I=1}^2 X_I^{-2} \left(\frac{1}{2} F^I_a{}^c F^I_{bc} - \frac{1}{8} F^{Icd} F^I_{cd} g_{ab} \right) \\ & + X_1^2 X_2^2 \left(\frac{1}{12} F_a{}^{cde} F_{bcde} - \frac{1}{96} F^{cdef} F_{cdef} g_{ab} \right) \\ & + g^2(8X_1 X_2 + 4X_1^{-1} X_2^{-2} + 4X_1^{-2} X_2^{-1} - X_1^{-4} X_2^{-4}) g_{ab}.\end{aligned}\quad (3.3)$$

The remaining field equations are

$$\begin{aligned}\square\varphi_1 = & \frac{1}{2\sqrt{10}} \sum_{I=1}^2 X_I^{-2} F^{Iab} F^I_{ab} - \frac{1}{12\sqrt{10}} X_1^2 X_2^2 F^{abcd} F_{abcd}, \\ & + \frac{8}{\sqrt{10}} g^2(4X_1 X_2 - 3X_1^{-1} X_2^{-2} - 3X_1^{-2} X_2^{-1} + 2X_1^{-4} X_2^{-4}), \\ \square\varphi_2 = & \frac{1}{2\sqrt{2}} (X_1^{-2} F^{1ab} F^1_{ab} - X_2^{-2} F^{2ab} F^2_{ab}) + 4\sqrt{2} g^2 (X_1^{-1} X_2^{-2} - X_1^{-2} X_2^{-1}), \\ d(X_1^{-2} \star F_{(2)}^1) = & F_{(2)}^2 \wedge F_{(4)}, \\ d(X_2^{-2} \star F_{(2)}^2) = & F_{(2)}^1 \wedge F_{(4)}, \\ d(X_1^2 X_2^2 \star F_{(4)}) = & 2gF_{(4)} + F_{(2)}^1 \wedge F_{(2)}^2.\end{aligned}\quad (3.4)$$

Once the field equations arising from the Lagrangian are satisfied, there is also a self-duality condition to impose, which can be stated by including a two-form potential $A_{(2)}$ and defining

$$F_{(3)} = dA_{(2)} - \frac{1}{2} A_{(1)}^1 \wedge dA_{(1)}^2 - \frac{1}{2} A_{(1)}^2 \wedge dA_{(1)}^1. \quad (3.5)$$

The self-duality equation is

$$X_1^2 X_2^2 \star F_{(4)} = 2gA_{(3)} - F_{(3)}. \quad (3.6)$$

If we truncate to solutions with $X = X_1 = X_2 = e^{-\varphi_1/\sqrt{10}}$, $\varphi_2 = 0$ and $A_{(1)} = A_{(1)}^1 = A_{(1)}^2$, then the bosonic field equations can be obtained from the Lagrangian

$$\begin{aligned}\mathcal{L}_7 = & R \star 1 - \frac{1}{2} \star d\varphi_1 \wedge d\varphi_1 - X^{-2} \star F_{(2)} \wedge F_{(2)} - \frac{1}{2} X^4 \star F_{(4)} \wedge F_{(4)} \\ & + 2g^2(8X^2 + 8X^{-3} - X^{-8}) \star 1 + F_{(2)} \wedge F_{(2)} \wedge A_{(3)} - gF_{(4)} \wedge A_{(3)},\end{aligned}\quad (3.7)$$

where $F_{(2)} = dA_{(1)}$. Setting $g = 0$, we recover the Lagrangian of (2.5). The self-duality equation becomes

$$X^4 \star F_{(4)} = 2gA_{(3)} - dA_{(2)} + F_{(2)} \wedge A_{(1)}. \quad (3.8)$$

3.1 Black hole solutions

Before presenting the new black hole solution, we first review the known black hole solutions in seven dimensions. The starting point for rotating black holes in higher dimensions is the Myers–Perry black hole [20], which generalizes the Kerr solution of four dimensions. In arbitrary dimensions, there is a generalization to include a cosmological constant [27, 28], and, as discussed previously, in the context of toroidally compactified supergravity, a generalization to include charges [17]. In seven dimensions, viewed in the context of $U(1)^2$ gauged supergravity, these respectively provide black hole solutions with three arbitrary rotation parameters, no charges, and arbitrary gauge coupling constant g ; and with three arbitrary rotation parameters, two arbitrary charges, and zero gauge coupling.

Specific to seven dimensional gauged supergravity, a non-rotating black hole with two independent $U(1)$ charges and arbitrary gauge coupling constant was given in [29, 30]. A generalization to include three equal angular momenta, as well as the two independent $U(1)$ charges and arbitrary gauge coupling, was obtained in [11]. The solution involves the Fubini–Study metric on \mathbb{CP}^2 ,

$$d\Sigma_2^2 = d\xi^2 + \frac{1}{4} \sin^2 \xi (\sigma_1^2 + \sigma_2^2) + \frac{1}{4} \sin^2 \xi \cos^2 \xi \sigma_3^2, \quad (3.9)$$

where σ_i are left-invariant one-forms on $SU(2)$ that satisfy $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$, for which the Kähler form is $J = \frac{1}{2}d\sigma$ with $\sigma = d\tau + \frac{1}{2}\sin^2 \xi \sigma_3$. A simplification occurs if the two $U(1)$ charges are set equal. Compared with [11], we perform the coordinate changes $t \rightarrow (1 - ag)t$, followed by $\tau \rightarrow \tau - gt$, so $\sigma \rightarrow \sigma - gdt$, and so the solution is written as

$$\begin{aligned} ds^2 &= H^{2/5} \left[-\frac{V - 2m}{H^2 \Xi^2 (r^2 + a^2)^2} (dt - a\sigma)^2 + \frac{(r^2 + a^2)^2}{V - 2m} dr^2 + \frac{r^2 + a^2}{\Xi} d\Sigma_2^2 \right. \\ &\quad \left. + \frac{a^2}{\Xi^2 r^2} \left((1 + g^2 r^2) dt - \frac{r^2 + a^2}{a} \sigma - \frac{2ms^2(1 + ag)}{H(r^2 + a^2)^2} (dt - a\sigma) \right)^2 \right], \\ X &= H^{-1/5}, \quad A_{(1)} = \frac{2msc}{H\Xi(r^2 + a^2)^2} (dt - a\sigma), \quad A_{(2)} = \frac{2ms^2(1 + ag)}{H\Xi(r^2 + a^2)^2} dt \wedge a\sigma, \\ A_{(3)} &= \frac{2ms^2}{\Xi^2(r^2 + a^2)^2} [a(\sigma - ag^2 dt) - ag(dt - a\sigma)] \wedge (r^2 + a^2)J, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} V &= \frac{(1 + g^2 r^2)(r^2 + a^2)^3}{r^2} + 2ms^2 g^2 (2r^2 + 3a^2) - \frac{4ms^2 g a^3}{r^2} + \frac{(2ms^2)^2 g^2}{r^2}, \\ H &= 1 + \frac{2ms^2}{(r^2 + a^2)^2}, \quad s = \sinh \delta, \quad c = \cosh \delta, \quad \Xi = 1 - a^2 g^2. \end{aligned} \quad (3.11)$$

The time coordinate t that appears here is canonically normalized and matches the t of, for example, [23].

Guided by the structure of the solutions we have just discussed, we are in a position to obtain a new seven dimensional solution with three independent angular momenta and equal charges. In particular, we are helped by the simple form of the sevenbeins in terms of which the previously known solutions may be written.

The new solution is

$$\begin{aligned}
ds^2 = & H^{2/5} \left\{ \frac{(r^2 + y^2)(r^2 + z^2)}{R} dr^2 + \frac{(r^2 + y^2)(y^2 - z^2)}{Y} dy^2 + \frac{(r^2 + z^2)(z^2 - y^2)}{Z} dz^2 \right. \\
& - \frac{R}{H^2(r^2 + y^2)(r^2 + z^2)} \mathcal{A}^2 \\
& + \frac{Y}{(r^2 + y^2)(y^2 - z^2)} \left[dt' + (z^2 - r^2)d\psi_1 - r^2 z^2 d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \right]^2 \\
& + \frac{Z}{(r^2 + z^2)(z^2 - y^2)} \left[dt' + (y^2 - r^2)d\psi_1 - r^2 y^2 d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \right]^2 \\
& + \frac{a_1^2 a_2^2 a_3^2}{r^2 y^2 z^2} \left[dt' + (y^2 + z^2 - r^2)d\psi_1 + (y^2 z^2 - r^2 y^2 - r^2 z^2)d\psi_2 - r^2 y^2 z^2 d\psi_3 \right. \\
& \left. \left. - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \left(1 + \frac{g y^2 z^2}{a_1 a_2 a_3} \right) \mathcal{A} \right]^2 \right\}, \\
X = & H^{-1/5}, \quad A_{(1)} = \frac{2msc}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A}, \\
A_{(3)} = & q a_1 a_2 a_3 [d\psi_1 + (y^2 + z^2)d\psi_2 + y^2 z^2 d\psi_3] \\
& \wedge \left(\frac{1}{(r^2 + y^2)z} dz \wedge (d\psi_1 + y^2 d\psi_2) + \frac{1}{(r^2 + z^2)y} dy \wedge (d\psi_1 + z^2 d\psi_2) \right) \\
& - qg \mathcal{A} \wedge \left(\frac{z}{r^2 + y^2} dz \wedge (d\psi_1 + y^2 d\psi_2) + \frac{y}{r^2 + z^2} dy \wedge (d\psi_1 + z^2 d\psi_2) \right), \quad (3.12)
\end{aligned}$$

where

$$\begin{aligned}
R = & \frac{1 + g^2 r^2}{r^2} \prod_{i=1}^3 (r^2 + a_i^2) + qg^2(2r^2 + a_1^2 + a_2^2 + a_3^2) - \frac{2qga_1 a_2 a_3}{r^2} + \frac{q^2 g^2}{r^2} - 2m, \\
Y = & \frac{1 - g^2 y^2}{y^2} \prod_{i=1}^3 (a_i^2 - y^2), \quad Z = \frac{1 - g^2 z^2}{z^2} \prod_{i=1}^3 (a_i^2 - z^2), \\
\mathcal{A} = & dt' + (y^2 + z^2)d\psi_1 + y^2 z^2 d\psi_2, \\
H = & 1 + \frac{q}{(r^2 + y^2)(r^2 + z^2)}, \quad q = 2ms^2, \quad s = \sinh \delta, \quad c = \cosh \delta. \quad (3.13)
\end{aligned}$$

The two-form potential is

$$\begin{aligned}
A_{(2)} = & \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \wedge \\
& \left(dt' + \sum_i a_i^2 (g^2 dt' + d\psi_1) + \sum_{i < j} a_i^2 a_j^2 (g^2 d\psi_1 + d\psi_2) + a_1^2 a_2^2 a_3^2 (g^2 d\psi_2 + d\psi_3) \right. \\
& \left. - g^2 (y^2 + z^2) dt' - g^2 y^2 z^2 d\psi_1 + a_1 a_2 a_3 g [d\psi_1 + (y^2 + z^2)d\psi_2 + y^2 z^2 d\psi_3] \right). \quad (3.14)
\end{aligned}$$

It is straightforward to verify on a computer that the above solution does indeed satisfy the field equations. The natural choice of sevenbeins facilitates the computations, including those of the metric determinant and of the metric inverse.

The structure of this seven dimensional solution is analogous to the five dimensional gauged supergravity solution of [7] with two equal U(1) charges and the third U(1) charge equal to

a particular value. However, the solution could also be thought of as analogous to the five dimensional black hole of minimal gauged supergravity, for which all three $U(1)$ charges are set equal [8].

For computing thermodynamical quantities, as advocated in [32], we use angular velocities measured with respect to a non-rotating frame at infinity and move to Boyer–Lindquist coordinates using the coordinate change [23]

$$\begin{aligned} t &= t' + (a_1^2 + a_2^2 + a_3^2)\psi_1 + (a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2)\psi_2 + a_1^2 a_2^2 a_3^2 \psi_3, \\ \frac{\phi_i}{a_i} &= g^2 t' + \psi_1 + \sum_{j \neq i} a_j^2 (g^2 \psi_1 + \psi_2) + \prod_{j \neq i} a_j^2 (g^2 \psi_2 + \psi_3). \end{aligned} \quad (3.15)$$

It is routine to perform the coordinate change, although it may be helpful to note here that the two-form potential is

$$\begin{aligned} A_{(2)} &= \frac{q(a_1 + a_2 a_3 g)}{H(r^2 + y^2)(r^2 + z^2)} \\ &\quad \left(\frac{(1 - g^2 y^2)(1 - g^2 z^2)}{\Xi_1 \Xi_2 \Xi_3} dt \wedge \mu_1^2 d\phi_1 + \frac{g(a_3^2 - a_2^2)\mu_2^2 \mu_3^2}{\Xi_2 \Xi_3} d\phi_2 \wedge d\phi_3 \right) + \text{cyclic}, \end{aligned} \quad (3.16)$$

where there are two additional terms by cycling the indices 1, 2, 3, and, from (2.7),

$$\mu_i^2 = \frac{(a_i^2 - y^2)(a_i^2 - z^2)}{\prod_{j \neq i} (a_i^2 - a_j^2)}. \quad (3.17)$$

We have also denoted $\Xi_i = 1 - a_i^2 g^2$, which are positive so that the signature is correct. In these $(t, r, y, z, \phi_1, \phi_2, \phi_3)$ coordinates, the metric determinant is

$$\det(g_{ab}) = \frac{H^{4/5} r^2 y^2 z^2 (r^2 + y^2)^2 (r^2 + z^2)^2 (y^2 - z^2)^2}{(a_1^2 - a_2^2)^2 (a_2^2 - a_3^2)^2 (a_3^2 - a_1^2)^2}. \quad (3.18)$$

3.2 Curvature singularities

The presentation of the higher dimensional Kerr–NUT–AdS solution of [23] gives a simple orthonormal basis with which one may compute the curvature, as done in [31]. The structure of the curvature singularities is similar to that discussed in [20] for the Myers–Perry solution. If any rotation parameter a_i vanishes, then there is a curvature singularity at $r = 0$. For the general case in which none of the rotation parameters a_i vanish, we might be worried about singular behaviour at $r = 0$ since there are singularities in the vielbein components there. However, it turns out that all the curvature components $R_{\mu\nu\rho\sigma}$ in this orthonormal basis are well-behaved at $r = 0$ provided no a_i vanishes, and also the metric components g_{ab} and inverse metric components g^{ab} are well-behaved there.

For the seven dimensional solution (3.12) we have obtained, we may perform the coordinate change $u = r^2$, and it turns out that the metric components g_{ab} and inverse metric components g^{ab} , as well as the metric determinant $\det(g_{ab})$, are all non-singular at $u = 0$. A simple choice of orthonormal frame may be read off from the way the metric is presented, and includes

$$\begin{aligned} e_0 &= H^{-4/5} \frac{\sqrt{R}}{\sqrt{(r^2 + y^2)(r^2 + z^2)}} \mathcal{A}, \\ e_6 &= \frac{H^{1/5} a_1 a_2 a_3}{r y z} \left[dt' + (y^2 + z^2 - r^2) d\psi_1 + (y^2 z^2 - r^2 y^2 - r^2 z^2) d\psi_2 - r^2 y^2 z^2 d\psi_3 \right. \\ &\quad \left. - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \left(1 + \frac{g y^2 z^2}{a_1 a_2 a_3} \right) \mathcal{A} \right]. \end{aligned} \quad (3.19)$$

However, in the $(t', r, y, z, \psi_1, \psi_2, \psi_3)$ coordinates, regardless of any coordinate change $u = r^2$, there remains singular behaviour in the sevenbein components e_0 and e_6 at $r = 0$. These singularities conspired to cancel each other out when forming metric components, but one might still be wary of a curvature singularity there, as some orthonormal components of the curvature $R_{\mu\nu\rho\sigma}$ diverge. However, such an apparent singularity is caused by a bad choice of orthonormal frame. We have, in these coordinates, orthonormal frame components

$$\begin{aligned} e_0 &= \left(1 + \frac{q}{y^2 z^2}\right)^{-4/5} \frac{a_1 a_2 a_3 - qg}{ryz} \mathcal{A} + \mathcal{O}(r), \\ e_6 &= \left(1 + \frac{q}{y^2 z^2}\right)^{-4/5} \frac{a_1 a_2 a_3 - qg}{ryz} \mathcal{A} + \mathcal{O}(r), \end{aligned} \quad (3.20)$$

so there is a degeneracy in the orthonormal frame as $r \rightarrow 0$. We may remove the degeneracy by changing orthonormal frames and performing local Lorentz boosts with arbitrarily large rapidity as $r \rightarrow 0$. For example, we could replace e_0 and e_6 in favour of

$$\begin{aligned} e'_0 &= \frac{1}{r} e_0 - \frac{\sqrt{1-r^2}}{r} e_6 = \mathcal{O}(r^0), \\ e'_6 &= \frac{1}{r} e_6 - \frac{\sqrt{1-r^2}}{r} e_0 = \mathcal{O}(r^0), \end{aligned} \quad (3.21)$$

leaving e_1, \dots, e_5 unchanged. The new inverse sevenbein components e_μ^a are also well-behaved at $r = 0$. It follows that the curvature components $R_{\mu\nu\rho\sigma}$ in this orthonormal frame must be non-singular at $r = 0$, so the geometry is regular there. Using the new radial coordinate $u = r^2$, the metric may be extended to negative values of u , which may be thought of as extending to imaginary values of r . If one examines the orthonormal components of the Riemann tensor, then one finds negative powers of $r^2 + y^2$ and $r^2 + z^2$, and so there is a curvature singularity that extends out to a maximum radius given by $r^2 = -\min(a_1^2, a_2^2, a_3^2)$. There are also negative powers of $(r^2 + y^2)(r^2 + z^2) + q$ appearing in the curvature, which moves the curvature singularity further out for $q < 0$. If there is a horizon at $r = r_0$ or some minimum radius to the geometry at $r = r_0$, then a naked singularity is avoided if $r_0^2 + a_i^2 > 0$ for each i and $(r_0^2 + a_i^2)(r_0^2 + a_j^2) + q > 0$ for each $i \neq j$. From now on, we assume that the parameters of the solution are chosen so that the outermost curvature singularity is hidden behind a horizon.

3.3 Thermodynamics

The outer black hole horizon is located at the largest root of $R(r)$, say at $r = r_+$. Its angular velocities are constant over the horizon and are obtained from the Killing vector $\ell = \partial/\partial t + \sum_i \Omega_i \partial/\partial \phi_i$ that becomes null on the horizon. The surface gravity κ , also constant over the horizon, is given by $\ell^b \nabla_b \ell^a = \kappa \ell^a$ evaluated on the horizon. The horizon area is obtained from integrating the square root of the determinant of the induced metric on a time slice of the horizon,

$$\det g_{(y,z,\phi_1,\phi_2,\phi_3)}|_{r=r_+} = \frac{[\prod_i (r_+^2 + a_i^2) + q(r_+^2 - a_1 a_2 a_3 g)]^2 y^2 z^2 (y^2 - z^2)^2}{\Xi_1^2 \Xi_2^2 \Xi_3^2 (a_1^2 - a_2^2)^2 (a_2^2 - a_3^2)^2 (a_3^2 - a_1^2)^2 r_+^2}. \quad (3.22)$$

Bearing in mind that the radial coordinate may be analytically continued to negative values of r^2 , we should demand that $r_+^2 > 0$, so the horizon area, or equivalently the entropy, is real.

We take the temperature to be $T = \kappa/2\pi$ and the entropy to be one quarter of the horizon area. The angular momenta are given by the Komar integrals

$$J_i = \frac{1}{16\pi} \int_{S^5} \star dK_i, \quad (3.23)$$

where K_i is the one-form obtained from the Killing vector $\partial/\partial\phi_i$. Using the Killing vector ℓ that becomes null on the horizon, we obtain the electrostatic potential $\Phi = \ell \cdot A_{(1)}$ evaluated on the horizon, over which it is constant. The conserved Page electric charge is

$$Q = \frac{1}{8\pi} \int_{S^5} (X^{-2} \star F_{(2)} - F_{(2)} \wedge A_{(3)}), \quad (3.24)$$

although for our solution there is no contribution from the $F_{(2)} \wedge A_{(3)}$ term; our normalization factor of $1/8\pi$ rather than $1/16\pi$ arises from using the canonical normalization for two separate $U(1)$ fields and then setting them equal.

One finds that $TdS + \sum_i \Omega_i dJ_i + \Phi dQ$ is an exact differential, and so we may integrate the first law of black hole mechanics,

$$dE = TdS + \sum_i \Omega_i dJ_i + \Phi dQ, \quad (3.25)$$

to obtain an expression for the thermodynamic mass E . There are various other methods of obtaining the energy of an asymptotically AdS spacetime that we do not pursue here, but discussion of how various methods are applied to computing the conserved charges of AdS black hole solutions may be found in [32, 33] for example.

In summary, we find the thermodynamical quantities

$$\begin{aligned} E &= \frac{\pi^2}{8\Xi_1\Xi_2\Xi_3} \left[\sum_i \frac{2m}{\Xi_i} - m + \frac{5q}{2} + \frac{q}{2} \sum_i \left(\sum_{j \neq i} \frac{2\Xi_j}{\Xi_i} - \Xi_i - \frac{2(1 + 2a_1a_2a_3g^3)}{\Xi_i} \right) \right], \\ T &= \frac{(1 + g^2r_+^2)r_+^2 \sum_i \prod_{j \neq i} (r_+^2 + a_j^2) - \prod_i (r_+^2 + a_i^2) + 2q(g^2r_+^4 + ga_1a_2a_3) - q^2g^2}{2\pi r_+ [(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 + a_3^2) + q(r_+^2 - a_1a_2a_3g)]}, \\ S &= \frac{\pi^3 [(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 + a_3^2) + q(r_+^2 - a_1a_2a_3g)]}{4\Xi_1\Xi_2\Xi_3r_+}, \\ \Omega_i &= \frac{a_i [(1 + g^2r_+^2) \prod_{j \neq i} (r_+^2 + a_j^2) + qg^2r_+^2] - q \prod_{j \neq i} a_j g}{(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 + a_3^2) + q(r_+^2 - a_1a_2a_3g)}, \\ J_i &= \frac{\pi^2 m [a_i c^2 - s^2 g (\prod_{j \neq i} a_j + a_i \sum_{j \neq i} a_j^2 g + a_1a_2a_3a_i g^2)]}{4\Xi_1\Xi_2\Xi_3\Xi_i}, \\ \Phi &= \frac{2mscr_+^2}{(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 + a_3^2) + q(r_+^2 - a_1a_2a_3g)}, \\ Q &= \frac{\pi^2 msc}{\Xi_1\Xi_2\Xi_3}. \end{aligned} \quad (3.26)$$

The Gibbs free energy, $G = E - TS - \sum_i \Omega_i J_i - \Phi Q$, is

$$G = \frac{\pi^2}{16\Xi_1\Xi_2\Xi_3r_+^2} \left\{ (1 - g^2r_+^2) \prod_i (r_+^2 + a_i^2) - 2qg^2r_+^4 - 2qa_1a_2a_3g \right. \\ \left. - q^2 \left[g^2 \left(\sum_i a_i^2r_+^4 - \sum_{i<j} a_i^2a_j^2r_+^2 - \prod_i a_i^2 \right) + a_1a_2a_3g(2g^2r_+^4 - 2r_+^2 + qg^2) \right. \right. \\ \left. \left. + g^2r_+^2(r_+^4 + q) \right] \left(\prod_i (r_+^2 + a_i^2) + q(r_+^2 - a_1a_2a_3g) \right)^{-1} \right\}. \quad (3.27)$$

One then obtains the grand canonical partition function, $Z_{\text{gc}} = \exp(-G/T)$.

For black holes with a large horizon radius compared to the AdS radius, there are universal thermodynamical predictions arising from conformal fluid mechanics via the AdS/CFT correspondence [34, 35]. We set $g = 1$ and take the limit $r_+ \gg 1$ keeping $k = q/r_+^4$ fixed. In this large black hole limit, after multiplying integrals by $1/g^5 G_7 = 16N^3/3\pi^2$, the thermodynamics is summarized by

$$T = \frac{r_+(3-k)}{2\pi}, \quad \Omega_i = a_i, \quad \Phi = 2\pi T \frac{\sqrt{k}}{3-k}, \quad \ln Z_{\text{gc}} = \frac{64\pi^6 N^3 T^5}{3 \prod_i (1 - \Omega_i^2)} \frac{(1+k)^2}{(3-k)^6}. \quad (3.28)$$

These agree with the fluid mechanical predictions, with the first corrections being $\mathcal{O}(1/r_+^2)$.

3.4 BPS limit

Conditions for a solution to be BPS were obtained in [36] from considering eigenvalues of the Bogomolny matrix. In our case, for which the two U(1) charges have been set equal, once appropriate signs of ϕ_i and $A_{(1)}$ have been chosen, a BPS solution satisfies

$$E + g \sum_i J_i - Q = 0. \quad (3.29)$$

It should be noted that there is a typographical error in [36] concerning conditions for the vanishing of the eigenvalues of the Bogomolny matrix. Specifically, equation (4.14) should be $e^{\delta_1+\delta_2} = 1 + 2/ag$, $(1 + 2/ag)^{-1}$, $1 - 2/3ag$, $(1 - 2/3ag)^{-1}$, since E and J are invariant under $\delta_i \rightarrow -\delta_i$, but $Q_i \rightarrow -Q_i$; therefore $e^{\delta_1+\delta_2} = 1 + 2/ag$ corresponds to $E - gJ - \sum_i Q_i = 0$ and $e^{\delta_1+\delta_2} = 1 - 2/3ag$ corresponds to $E + 3gJ - \sum_i Q_i = 0$.

The BPS condition is satisfied if

$$e^{2\delta} = 1 - \frac{2}{(a_1 + a_2 + a_3)g}, \quad (3.30)$$

which recovers the type A and type B conditions of [36] on setting the a_i equal up to signs. For δ to be real, we must have $(a_1 + a_2 + a_3)g < 0$ or $(a_1 + a_2 + a_3)g > 2$, along with the previous requirement that $-1 < a_i g < 1$. Equivalently, in a form that is more directly useful, the BPS constraint is

$$q = -\frac{2m}{(a_1 + a_2 + a_3)g(2 - a_1g - a_2g - a_3g)}. \quad (3.31)$$

The Killing vector

$$K = \frac{\partial}{\partial t} - g \sum_i \frac{\partial}{\partial \phi_i} \quad (3.32)$$

is the square of a Killing spinor ϵ , i.e. $K^a = \bar{\epsilon}\gamma^a\epsilon$. Because of its spinorial square root, from Fierz identities one may show that K is non-spacelike, and we have $g(K, K) = -f^2$, with

$$f = H^{-4/5} \left(1 + \frac{qg(1 - a_1g - a_2g - a_3g)^2(r_h^2 + y^2)(r_h^2 + z^2)}{\Xi_{1-}\Xi_{2-}\Xi_{3-}(a_1 + a_2)(a_2 + a_3)(a_3 + a_1)(r^2 + y^2)(r^2 + z^2)} \right), \quad (3.33)$$

where

$$r_h^2 = \frac{a_1a_2 + a_2a_3 + a_3a_1 - a_1a_2a_3g}{1 - a_1g - a_2g - a_3g}. \quad (3.34)$$

We have also used the definition $\Xi_{i\pm} = 1 \pm a_i g$.

The supersymmetric solutions generally preserve $\frac{1}{8}$ supersymmetry, although there can be supersymmetry enhancement if more than one eigenvalue of the Bogomolny matrix vanishes. We should recall that these eigenvalues are

$$\begin{aligned} \lambda_{i\pm} &= E + gJ_i - g \sum_{j \neq i} J_j \pm Q, \\ \lambda_{4\pm} &= E + g(J_1 + J_2 + J_3) \pm Q; \end{aligned} \quad (3.35)$$

the number of supersymmetries preserved is the number of zero eigenvalues. We have chosen conventions (3.29) such that $\lambda_{4-} = 0$, but it may be possible for some of the other eigenvalues to also vanish. Since the charge Q does not vanish for the BPS solutions, as follows from (3.30), apart from for AdS_7 itself, we see that at most four of the eight eigenvalues (3.35) of the Bogomolny matrix can vanish. The possibilities for enhanced supersymmetry are, up to permutations,

$\frac{1}{4}$ supersymmetric	$\frac{3}{8}$ supersymmetric	$\frac{1}{2}$ supersymmetric	
$\lambda_{1+} = 0$	$\lambda_{1+} = \lambda_{2+} = 0$	$\lambda_{1+} = \lambda_{2+} = \lambda_{3+} = 0$	
$\lambda_{1-} = 0$	$\lambda_{1+} = \lambda_{2-} = 0$	$\lambda_{1+} = \lambda_{2+} = \lambda_{3-} = 0$	(3.36)
	$\lambda_{1-} = \lambda_{2-} = 0$	$\lambda_{1+} = \lambda_{2-} = \lambda_{3-} = 0$	
		$\lambda_{1-} = \lambda_{2-} = \lambda_{3-} = 0$	

The supersymmetric solutions might therefore be $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$ supersymmetric. Given a BPS solution, the eigenvalue λ_{1+} vanishes if

$$a_1g = \frac{4 - (a_2 + a_3)g - 3(a_2^2 + a_3^2)g^2 - 2a_2a_3g^2 - a_2a_3(a_2 + a_3)g^3}{4 + (a_2 + a_3)g - (a_2^2 + a_3^2)g^2 + 2a_2a_3g^2 + a_2a_3(a_2 + a_3)g^3}. \quad (3.37)$$

The eigenvalue λ_{1-} vanishes if either of the following two conditions holds:

$$a_1g = \frac{1 - (a_2 + a_3)g - 3a_2a_3g^2}{3 + (a_2 + a_3)g - a_2a_3g^2}, \quad (3.38)$$

$$a_2 + a_3 = 0. \quad (3.39)$$

However, we must take into account the inequalities that the rotation parameters must satisfy: $-1 < a_i g < 1$ for the correct signature; from the BPS constraint (3.30), either $(a_1 + a_2 + a_3)g < 0$ or $(a_1 + a_2 + a_3)g > 2$; for a real horizon area and entropy, a horizon $r = r_0$ must have $r_0^2 > 0$; and conditions to avoid a naked singularity.

To investigate whether or not the spacetime suffers from the pathology of naked closed timelike curves (CTCs), we write the metric in the form

$$\begin{aligned} ds^2 = H^{2/5} & \left[\frac{(r^2 + y^2)(r^2 + z^2)}{R} dr^2 + \frac{(r^2 + y^2)(y^2 - z^2)}{Y} dy^2 + \frac{(r^2 + z^2)(z^2 - y^2)}{Z} dz^2 \right. \\ & - \frac{r^2 y^2 z^2 R Y Z}{H^2 \prod_{i < j} (a_i^2 - a_j^2)^2 B_1 B_2 B_3} dt^2 + B_3 (d\phi_3 + v_{32} d\phi_2 + v_{31} d\phi_1 + v_{30} dt)^2 \\ & \left. + B_2 (d\phi_2 + v_{21} d\phi_1 + v_{20} dt)^2 + B_1 (d\phi_1 + v_{10} dt)^2 \right], \end{aligned} \quad (3.40)$$

so that the periodic ϕ_i coordinates have been separated from a dt^2 term. We have used (3.18) in writing this form of the metric, but for now do not require any details of the additional functions introduced, which one can straightforwardly obtain. However, it is worth noting that the functions B_i may be expressed using determinants of parts of the metric involving only $d\phi_i$, namely

$$B_3 = H^{-2/5} g_{\phi_3 \phi_3}, \quad B_2 = H^{-2/5} \frac{\det g_{(\phi_2, \phi_3)}}{g_{\phi_3 \phi_3}}, \quad B_1 = H^{-2/5} \frac{\det g_{(\phi_1, \phi_2, \phi_3)}}{\det g_{(\phi_2, \phi_3)}}. \quad (3.41)$$

There are CTCs if any B_i is negative; the determinants appear in the expressions for B_i as a manifestation of the standard result that a quadratic form is positive if and only if each of the leading minors is positive. From the g_{tt} coefficient, we have

$$\begin{aligned} -f^2 = & -\frac{r^2 y^2 z^2 R Y Z}{H^2 \prod_{i < j} (a_i^2 - a_j^2)^2 B_1 B_2 B_3} + B_3 [g(1 + v_{32} + v_{31}) - v_{30}]^2 \\ & + B_2 [g(1 + v_{21}) - v_{20}]^2 + B_1 (g - v_{10})^2. \end{aligned} \quad (3.42)$$

Since $R = 0$ at the horizon and the left hand side is negative semi-definite, we generally have some B_i negative near the horizon, and so the solution generally possesses naked CTCs. There are, however, two special cases for which naked CTCs do not occur, which we now discuss.

3.4.1 Supersymmetric black holes

One way to obtain solutions free from naked CTCs is to demand that $f = 0$ at the horizon, which leads to the further condition

$$q = -\frac{\Xi_1 - \Xi_2 - \Xi_3 - (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)}{(1 - a_1 g - a_2 g - a_3 g)^2 g}. \quad (3.43)$$

We then have the simplification

$$f = H^{-4/5} \left(1 - \frac{(r_0^2 + y^2)(r_0^2 + z^2)}{(r^2 + y^2)(r^2 + z^2)} \right), \quad (3.44)$$

where $r_0^2 = r_h^2$ is given by (3.34) and denotes the location of a horizon. At the horizon, we have $R = 0$, and at a horizon with non-zero area, $B_1 B_2 B_3 \neq 0$. Therefore from (3.42), if $f = 0$ at the horizon, then each of $g(1 + v_{32} + v_{31}) - v_{30}$, $g(1 + v_{21}) - v_{20}$ and $g - v_{10}$ must vanish at the horizon. Differentiating (3.42) with respect to r , we see that R' also vanishes at the horizon, so the radial function R must possess a double root, and we find that it takes the form

$$R = \frac{(r^2 - r_0^2)^2}{r^2} \left(g^2 r^4 + [1 + (a_1^2 + a_2^2 + a_3^2)g^2 + 2g^2 r_0^2] r^2 + \frac{(a_1 a_2 a_3 - qg)^2}{r_0^4} \right). \quad (3.45)$$

There should be no other horizons outside $r = r_0$ to avoid naked CTCs, and this is guaranteed by positive r_0^2 .

We then need to verify that each B_i is non-negative outside the horizon, which will place constraints on the parameters in terms of inequalities. The expressions for B_i are rather complicated, and so we do not provide a full analysis. However, if we choose each a_i to be positive but otherwise arbitrary, then one may verify that by taking g negative but with $|g|$ sufficiently small we obtain an example of a solution free of naked CTCs.

The thermodynamical quantities simplify to

$$\begin{aligned}
E &= -\frac{\pi^2 \prod_{k<l} (a_k + a_l) [\sum_i \Xi_i + \sum_{i<j} \Xi_i \Xi_j - (1 + a_1 a_2 a_3 g^3)(2 + \sum_i a_i g + \sum_{i<j} a_i a_j g^2)]}{8\Xi_{1+}^2 \Xi_{2+}^2 \Xi_{3+}^2 (1 - a_1 g - a_2 g - a_3 g)^2 g}, \\
S &= -\frac{\pi^3 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)(a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g)}{4\Xi_{1+} \Xi_{2+} \Xi_{3+} (1 - a_1 g - a_2 g - a_3 g)^2 g r_0}, \\
J_i &= -\frac{\pi^2 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1) [a_i - (a_i^2 + 2a_i \sum_{j \neq i} a_j + \prod_{j \neq i} a_j)g + a_1 a_2 a_3 g^2]}{8\Xi_{1+} \Xi_{2+} \Xi_{3+} \Xi_{i+} (1 - a_1 g - a_2 g - a_3 g)^2 g}, \\
Q &= -\frac{\pi^2 (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)}{2\Xi_{1+} \Xi_{2+} \Xi_{3+} (1 - a_1 g - a_2 g - a_3 g)g}, \\
T &= 0, \quad \Omega_i = -g, \quad \Phi = -1.
\end{aligned} \tag{3.46}$$

If $a = a_1 = -a_2 = -a_3$, then $r_0^2 = -a^2$ and so $f = H^{-4/5}$. Therefore $-f^2 = -H^{-8/5}$ is negative definite, and so there cannot be supersymmetric black holes. The absence of supersymmetric black holes in this case, and analogously in five dimensional supergravity with rotation parameters $a_1 = -a_2$, was noted in the analysis of [36], there referred to as type A.

We now consider whether the supersymmetric black holes can preserve more than $\frac{1}{8}$ supersymmetry. First, we consider the vanishing of λ_{1+} , in which case (3.37) holds. Then $r_0^2 < 0$, so there are no such solutions. There are two possibilities (3.38, 3.39) for the eigenvalue λ_{1-} vanishing, however the first, (3.38), is trivial for supersymmetric black holes and satisfied only for AdS_7 , so we consider the second, (3.39). Then $(a_1 + a_2 + a_3)g < 2$, but then it is not possible to satisfy both $(a_1 + a_2 + a_3)g < 0$ for the BPS constraint and $r_0^2 > 0$, so again there are no such solutions.

3.4.2 Topological solitons

The second way of avoiding naked CTCs is to demand that $B_1 B_2 B_3$ also vanishes at the outermost root of R , so then the spacetime has some minimum radius at which the geometry remains smooth, giving rise to a topological soliton. From (3.22) and the expression for the radial function R , we find that topological solitons occur if the BPS constraint is supplemented by

$$\begin{aligned}
q &= \frac{\prod_i \Xi_{i-} [a_i - (a_i^2 + 2a_i \sum_{j \neq i} a_j + \prod_{j \neq i} a_j)g + a_1 a_2 a_3 g^2]}{(1 - a_1 g - a_2 g - a_3 g)^4 g} \\
&= \frac{\prod_i \Xi_{i-} (a_i - g r_h^2)}{(1 - a_1 g - a_2 g - a_3 g)g}.
\end{aligned} \tag{3.47}$$

The geometry ends at $r = r_0$ with

$$r_0^2 = -\frac{(a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g)^2 g^2}{(1 - a_1 g - a_2 g - a_3 g)^2} = -g^2 r_h^4. \tag{3.48}$$

Since there is no horizon, it is not necessary to demand that $r_0^2 > 0$, unlike the case of supersymmetric black holes, where it was needed to ensure that the horizon area and entropy were real. Since our expression for r_0^2 is not positive, it is convenient to define a new radial coordinate $\hat{r}^2 = r^2 - r_0^2$, which takes values $0 \leq \hat{r} < \infty$.

In general, there is a conical singularity at $\hat{r} = 0$, as may be seen from a relevant part of the metric,

$$\begin{aligned} ds^2 &= H^{2/5} \left(\frac{(r^2 + y^2)(r^2 + z^2)}{R} dr^2 + B_1 (d\phi_1 + v_{10} dt)^2 \right) + \dots \\ &= H^{2/5} (r_0) (r_0^2 + y^2) (r_0^2 + z^2) \\ &\quad \left(\frac{(1 - a_1 g - a_2 g - a_3 g)^4 d\hat{r}^2}{\Xi_{1-} \Xi_{2-} \Xi_{3-} C (a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g)^2} \right. \\ &\quad \left. + \frac{C \Xi_{2-} \Xi_{3-} (a_2 + a_3 a_1 g)^2 (a_3 + a_1 a_2 g)^2 \hat{r}^2 (d\phi_1 + v_{10} dt)^2}{\Xi_{1+}^2 \Xi_{1-} (1 - a_1 g - a_2 g - a_3 g)^6 (r_0^2 + a_2^2)^2 (r_0^2 + a_3^2)^2} \right) + \dots, \end{aligned} \quad (3.49)$$

where

$$\begin{aligned} C &= 1 - 5 \sum_i a_i g + 7 \sum_i a_i^2 g^2 + 19 \sum_{i < j} a_i a_j g^2 - 3 \sum_i a_i^3 g^3 - 19 \sum_{i \neq j} a_i^2 a_j g^3 \\ &\quad - 51 a_1 a_2 a_3 g^3 + 5 \sum_{i \neq j} a_i^3 a_j g^4 + 14 \sum_{i < j} a_i^2 a_j^2 g^4 + 39 a_1 a_2 a_3 \sum_i a_i g^4 \\ &\quad - 3 a_1 a_2 a_3 \sum_i a_i^2 g^5 - 14 a_1 a_2 a_3 \sum_{i < j} a_i a_j g^5 + 4 a_1^2 a_2^2 a_3^2 g^6. \end{aligned} \quad (3.50)$$

To ensure that there is no conical singularity at $\hat{r} = 0$, since ϕ_1 has period 2π , we need the quantization condition

$$\left(\frac{\Xi_{2-} \Xi_{3-} C (a_2 + a_3 a_1 g) (a_3 + a_1 a_2 g) (a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3 g)}{\Xi_{1+} (1 - a_1 g - a_2 g - a_3 g)^5 (r_0^2 + a_2^2) (r_0^2 + a_3^2)} \right)^2 = 1. \quad (3.51)$$

The supersymmetric topological solitons of [36] are given by $a = a_1 = a_2 = a_3$, there known as type B, and $a = a_1 = -a_2 = -a_3$, there known as type A. For both cases, the quantization condition (3.51) cannot hold for any rotation parameter with $-1 < ag < 1$ and with either $(a_1 + a_2 + a_3)g < 0$ or $(a_1 + a_2 + a_3)g > 2$. One could instead consider making ϕ_1 have period $2\pi/k$ instead, for some positive integer k , leading to solutions that are asymptotically $\text{AdS}_7/\mathbb{Z}_k$, in which case it becomes possible for a quantization condition to hold with $-1 < ag < 1$. We now consider whether the supersymmetric topological solitons can preserve more than $\frac{1}{8}$ supersymmetry, for example the case $a = a_1 = -a_2 = -a_3$ considered in [36] was shown to be $\frac{3}{8}$ supersymmetric. We should again check, as we previously did with the supersymmetric black holes, whether the rotation parameters lie within the allowed ranges to ensure the correct signature, so that the BPS constraint is satisfied, and so that there are no naked singularities. Additionally, we should check that we can find rotation parameters for which the quantization condition (3.51) holds. The eigenvalue λ_{1+} vanishes if (3.37) holds. It is possible to find rotation parameters with $-1 < a_2 g < 1$ and $-1 < a_3 g < 1$ so that $(a_1 + a_2 + a_3)g < 0$, but then it is not possible to satisfy $-1 < a_1 g < 1$. The other possibility of satisfying the BPS constraint, of $(a_1 + a_2 + a_3)g > 2$, does not occur. This argument could have been used instead to rule out $\lambda_{1+} = 0$ for the supersymmetric black holes, since it does not rely on any expression for r_0 . We next consider the two possibilities (3.38, 3.39) for the eigenvalue λ_{1-} vanishing. If we have (3.38), then we may choose $a_2 g$ and

$a_3 g$ so that the BPS constraint is satisfied through $(a_1 + a_2 + a_3)g < 0$. However, we then find that $r_0^2 + \min(a_2^2, a_3^2) < 0$, and so a naked singularity cannot be avoided. We therefore move on to the second possibility that leads to vanishing λ_{1-} , (3.39). Setting $a_3 = -a_1$ with a_2 independent gives $\frac{1}{4}$ supersymmetric topological solitons; we find that the rotation parameters may be chosen so that we have the correct signature, a smooth geometry, satisfy the BPS condition, and avoid naked CTCs. The case $a = a_1 = -a_2 = -a_3$, which preserves $\frac{3}{8}$ supersymmetry, as noted above, cannot be asymptotically AdS_7 , and satisfy both the BPS condition and the quantization condition.

4 Discussion

We have obtained a black hole solution of seven dimensional gauged supergravity with arbitrary angular momenta and equal $\text{U}(1)$ charges in the $\text{U}(1)^2$ truncation of the full $\text{SO}(5)$ gauge group, complementing the solution of the ungauged theory with arbitrary angular momenta and arbitrary charges [17], and the solution of the gauged theory with equal angular momenta and arbitrary charges [11]. It remains to find a general black hole solution of the gauged theory with arbitrary angular momenta and arbitrary charges.

We have demonstrated similarities between some black hole solutions of gauged supergravity theories in various dimensions in the case of certain combinations of charges being set equal. These may serve as a guide to obtaining general black hole solutions of four and five dimensional gauged supergravity with arbitrary angular momenta and arbitrary charges as well.

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